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A New Algebraic Compatibility Framework for the Liénard Differential Equation

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Abstract

The relentless search for new mathematical methodologies capable of addressing longstanding open problems has motivated the development of increasingly innovative analytical frameworks for nonlinear differential equations. Inspired by this perspective, this work introduces a new Algebraic Compatibility Operator for the Liénard differential equation, transforming the original nonlinear differential operator into an equivalent polynomial compatibility problem. Unlike classical approaches based on qualitative analysis and limit-cycle theory, the proposed methodology establishes a constructive algebraic framework for investigating rational solutions through finite polynomial systems. The theoretical formulation is supported by fundamental equivalence theorems and validated numerically using the classical Van der Pol equation, demonstrating consistency between the proposed algebraic formulation and the nonlinear dynamics of the original system. The obtained results provide a new mathematical perspective for the analytical investigation of the Liénard equation and establish a promising foundation for future developments involving nonlinear differential equations and related open mathematical problems.

Keywords: Liénard equation; Algebraic Compatibility Operator; Nonlinear Differential Equations; Polynomial Compatibility; Rational Solutions.

Introduction

In recent years, increasing attention has been devoted to the development of new mathematical methodologies for addressing longstanding open problems in pure and applied mathematics. Rather than relying exclusively on classical analytical techniques, several recent investigations have proposed alternative mathematical formulations capable of providing new perspectives for problems whose complete analytical solution remains unknown. Recent examples include new analytical frameworks for the three-dimensional Navier–Stokes equations [1], the Painlevé equivalence problem [2], historical and modern investigations on the Navier–Stokes Millennium Prize Problem [7, 9], current open questions concerning the Painlevé equations [8], and recent mathematical developments related to the Yang–Mills Millennium Problem [10]. Together, these studies illustrate a growing tendency in contemporary mathematics: the search for new mathematical structures capable of reformulating difficult differential problems from alternative analytical viewpoints.

Within this context, the Liénard differential equation occupies a central position in nonlinear analysis due to its numerous applications in nonlinear oscillations, electrical circuits, biological systems, mechanical vibrations and dynamical systems. The general Liénard equation,

$$y'' + f(y)y' + g(y) = 0,$$

includes several classical nonlinear models as particular cases, among them the celebrated Van der Pol oscillator. Despite its apparent simplicity, the general analytical treatment of nonlinear Liénard equations remains highly challenging, especially when rational or explicit analytical solutions are sought.

Most recent investigations concerning the Liénard equation have concentrated on the qualitative theory of nonlinear differential equations. Existing methodologies primarily investigate the existence, uniqueness and stability of limit cycles, periodic oscillations, invariant regions and qualitative properties of nonlinear trajectories. Although these approaches have produced important theoretical advances, comparatively little attention has been devoted to the development of constructive algebraic methodologies capable of reformulating the differential operator itself into an equivalent algebraic structure.

Motivated by this observation, the present work introduces an Algebraic Compatibility Operator for the Liénard equation. Instead of investigating qualitative properties of nonlinear trajectories, the proposed methodology transforms the original differential operator into an equivalent polynomial operator, allowing the nonlinear differential equation to be reformulated as a finite algebraic compatibility problem. This transformation establishes a constructive mathematical framework that connects nonlinear differential equations with polynomial algebra and symbolic computation.

The proposed methodology is subsequently applied to the classical Van der Pol equation, where both analytical developments and numerical simulations are employed to investigate the consistency of the proposed formulation. The obtained results indicate that the Algebraic Compatibility Operator preserves the fundamental nonlinear dynamics of the original system while providing a new algebraic interpretation of one of the most studied nonlinear differential equations in applied mathematics.

The remainder of this paper is organized as follows. Section 2 reviews the principal methodologies currently employed in the literature and presents the proposed algebraic formulation. Section 3 develops the mathematical theory, establishes the fundamental compatibility theorems and introduces the constructive algorithm. Numerical validation through the Van der Pol oscillator is then presented, followed by the discussion of the obtained results and the main conclusions.

General Objective

The general objective of this work is to introduce a new algebraic mathematical framework for the investigation of the nonlinear Liénard differential equation through the definition of an Algebraic Compatibility Operator capable of transforming the original differential operator into an equivalent polynomial compatibility problem. By replacing the direct analytical treatment of the differential equation with a constructive algebraic formulation, this work seeks to establish an alternative methodology for investigating rational solutions, expanding the mathematical tools currently available in the literature. The proposed framework is theoretically developed and numerically validated using the classical Van der Pol equation, providing a new analytical perspective that may contribute to future advances in the study of nonlinear differential equations and other longstanding open problems in pure and applied mathematics.

Methodology

Existing Methodologies for the Liénard Equation

The Liénard differential equation has been investigated for more than one century and remains one of the most important nonlinear models in the theory of differential equations. Nevertheless, most recent investigations concentrate on qualitative properties of the solutions, especially the existence, uniqueness and stability of limit cycles, instead of constructing new algebraic formulations capable of transforming the original nonlinear operator into an equivalent polynomial problem.

Villari and Zanolin [3] investigated generalized Liénard systems through qualitative analysis and topological methods.

Their methodology is based on phase-plane analysis and geometric arguments to establish sufficient conditions for the existence and qualitative behavior of periodic solutions. Although these results provide important information regarding nonlinear dynamics, the proposed methodology does not reformulate the differential operator itself, remaining restricted to qualitative investigations of the solution trajectories.

Zhang and Yan [4] proposed sufficient conditions for the uniqueness of limit cycles in generalized Liénard systems. Their approach combines analytical estimates together with geometric arguments to prove uniqueness under specific assumptions imposed on the nonlinear functions defining the equation. Consequently, the methodology provides important theoretical results regarding periodic behavior, but it is not intended to construct an algebraic representation of the differential operator nor a systematic procedure for determining analytical solutions.

Adjaï et al. [5] investigated periodic solutions of mixed Liénard-type equations using analytical techniques specifically developed for particular subclasses of nonlinear oscillators. Their methodology establishes conditions under which periodic solutions exist and studies the corresponding oscillatory behavior. However, the proposed framework remains associated with particular classes of equations and does not provide a general algebraic transformation applicable to arbitrary nonlinear Liénard equations.

Pérez-González, Torregrosa and Torres [6] generalized classical existence and uniqueness results for nonlinear Liénard equations involving nonlinear ϕ-Laplacian operators. Their methodology combines functional analysis, topological arguments and nonlinear operator theory to investigate periodic solutions. Although mathematically rigorous, the proposed formulation does not transform the original differential equation into an equivalent algebraic compatibility problem.

The methodologies reviewed above constitute important contributions to the qualitative theory of nonlinear differential equations. Nevertheless, within the literature analyzed in this work, no methodology based on the construction of an Algebraic Compatibility Operator capable of converting the Liénard differential operator into a polynomial compatibility equation was identified. Motivated by this observation, the present work proposes a different mathematical perspective based entirely on algebraic reformulation.

Proposed Methodology

The methodology proposed in this work is organized into six consecutive stages.
Initially, the general Liénard equation is represented through the proposed Algebraic Compatibility Operator. The differential operator is subsequently transformed into an equivalent polynomial operator by introducing a rational representation of the unknown function and eliminating all differential denominators through symbolic algebraic manipulation.

The resulting polynomial expression is expanded and converted into a finite compatibility equation by comparing coefficients associated with equal powers of the independent variable. The obtained algebraic system constitutes the mathematical foundation of the proposed compatibility criterion and of the constructive algorithm developed in the Results and Discussion section.

To evaluate the applicability of the proposed formulation, the classical Van der Pol equation was selected as the numerical benchmark. Numerical simulations were performed in Python in order to compare the analytical formulation with the nonlinear dynamics observed in the corresponding phase space.

Table 1 summarizes the principal computational parameters adopted throughout the numerical validation.

Tabela 1: Computational parameters adopted in the numerical simulations.

Parameter	Value
Numerical model	Classical Van der Pol equation
Parameter μ	1.0
Integration interval	$0 \leq t \leq 50$
Initial conditions	Multiple initial states
Numerical solver	solve_ivp
Programming language	Python 3
Scientific libraries	NumPy, SciPy, Matplotlib
Relative tolerance	10^{-9}
Absolute tolerance	10^{-9}
Outputs	Time series, phase portrait and limit cycle

Finally, the analytical formulation proposed in this work is compared with the numerical solutions obtained from the Van der Pol oscillator. The agreement between the algebraic formulation and the nonlinear dynamics provides a numerical consistency analysis for the proposed Algebraic Compatibility Operator and illustrates its potential as a new mathematical framework for investigating nonlinear Liénard equations.

Results and Discussion

2.3 Transformation of the Differential Operator into a Polynomial Operator

Consider the general Liénard differential equation

$$y'' + f(y)y' + g(y) = 0, \quad (1)$$

where $f(y)$ and $g(y)$ are sufficiently differentiable real-valued functions.

The main idea proposed in this work is to reformulate the differential operator associated with Equation (1) into an equivalent algebraic operator. Such a transformation replaces differential operations by polynomial compatibility relations, allowing the nonlinear differential equation to be analyzed through finite-dimensional algebraic techniques.

To this end, define the *Algebraic Compatibility Operator* associated with the Liénard equation by

$$\mathcal{A}_L[y] = \mathcal{T}(y'' + f(y)y' + g(y)), \quad (2)$$

where $\mathcal{T}$ denotes an algebraic transformation acting on the differential operator.
Assume that the unknown solution admits the rational representation

$$y(x) = \frac{P(x)}{Q(x)}, \quad (3)$$

where $P(x)$ and $Q(x)$ are real polynomials satisfying

$$Q(x) \neq 0.$$

The first derivative follows from the quotient rule,

$$y' = \frac{P'Q - PQ'}{Q^2}, \quad (4)$$

while the second derivative is obtained by differentiating Equation (4),

$$y'' = \frac{(P''Q - PQ'')Q^2 - 2(P'Q - PQ')QQ'}{Q^4}. \quad (5)$$

Substituting Equations (3), (4) and (5) into Equation (1) produces a rational differential expression whose denominator consists exclusively of powers of $Q(x)$.

The transformation $\mathcal{T}$ is defined as the algebraic elimination of every denominator by multiplication with the smallest common power of $Q(x)$. Consequently, the differential operator is converted into a polynomial operator,

$$\mathcal{A}_L[y] = \mathcal{P}(P, Q), \quad (6)$$

where $\mathcal{P}(P, Q)$ denotes a polynomial expression depending only on $P(x)$, $Q(x)$ and their polynomial derivatives.

Therefore, the original nonlinear differential equation is transformed into an equivalent polynomial relation, establishing the algebraic foundation of the proposed methodology.

2.4 Fundamental Algebraic Equivalence Theorem

The polynomial operator introduced in the previous section allows the Liénard equation to be interpreted as an algebraic compatibility problem. The following theorem establishes the mathematical equivalence between the original differential equation and the proposed polynomial formulation.

Theorem 1.

Let

$$y(x) = \frac{P(x)}{Q(x)},$$

where $P(x)$ and $Q(x)$ are real polynomials with $Q(x) \neq 0$. Then the Liénard equation

$$y'' + f(y)y' + g(y) = 0$$

is satisfied if and only if the associated polynomial operator satisfies

$$\mathcal{P}(P, Q) = 0. \tag{7}$$

Indeed, substituting the rational representation into the differential equation produces a rational expression whose denominator is a finite power of $Q(x)$. Since $Q(x)$ is assumed to be nonzero, multiplication by the common denominator preserves the set of admissible solutions. Consequently, the resulting numerator is a polynomial whose vanishing is equivalent to the vanishing of the original differential operator. Therefore, Equation (1) holds if and only if Equation (7) is satisfied.

Theorem 1 establishes the theoretical foundation of the proposed methodology. Instead of solving the nonlinear differential equation directly, the problem is reformulated as the determination of polynomial coefficients satisfying an algebraic compatibility condition. This equivalence transforms the original infinite-dimensional differential problem into a finite-dimensional algebraic framework, providing the basis for the constructive procedure developed in the following sections.

2.5 Construction of the Polynomial Compatibility Equation

The algebraic equivalence established in Theorem 1 allows the original Liénard equation to be reformulated as a polynomial compatibility problem. This reformulation constitutes the central mathematical result of the proposed methodology.

Starting from the rational representation

$$y(x) = \frac{P(x)}{Q(x)},$$

the corresponding first and second derivatives are substituted into Equation (1). Since every differential term is expressed as a rational function whose denominator is a finite power of $Q(x)$, multiplication by the least common denominator eliminates every rational component.

Consequently, the transformed equation assumes the polynomial form

$$\mathcal{P}(P, Q) = 0, \tag{8}$$

where

$$\mathcal{P}(P, Q) = \sum_{k=0}^{N} C_k x^k,$$

and each coefficient

$$C_k = C_k(a_0, \dots, a_m, b_0, \dots, b_n)$$

depends exclusively on the polynomial coefficients defining

$$P(x) = \sum_{i=0}^{m} a_i x^i,$$

and

$$Q(x) = \sum_{j=0}^{n} b_j x^j.$$

Equation (8) represents a polynomial identity. Therefore, it is satisfied if and only if every coefficient associated with each power of the independent variable vanishes simultaneously. This condition leads to the algebraic system

$$C_0 = C_1 = \dots = C_N = 0. \tag{9}$$

The resulting system constitutes the *Polynomial Compatibility Equation* associated with the Liénard differential equation.

Instead of solving the nonlinear differential equation directly, the determination of rational solutions is reduced to solving the finite algebraic system given by Equation (9).

This formulation establishes a direct correspondence between the differential operator and a polynomial compatibility condition, providing an entirely algebraic interpretation of the original nonlinear problem.

2.6 Necessary and Sufficient Compatibility Criterion

The polynomial compatibility equation established in the previous section naturally leads to a necessary and sufficient criterion for determining rational solutions of the Liénard equation. Rather than analyzing the nonlinear differential equation directly, the proposed methodology investigates the solvability of the associated finite algebraic system.

Theorem 2.
Let

$$P(x) = \sum_{i=0}^{m} a_i x^i, \qquad Q(x) = \sum_{j=0}^{n} b_j x^j,$$

with

$$Q(x) \neq 0.$$

Then the rational function

$$y(x) = \frac{P(x)}{Q(x)}$$

is a solution of the Liénard equation if and only if every coefficient of the polynomial compatibility equation satisfies

$$C_k = 0, \qquad k = 0,1,\dots,N.$$

Indeed, Theorem 1 establishes that the original differential equation is equivalent to the polynomial identity

$$\mathcal{P}(P,Q) = 0.$$

Since two polynomials are identical if and only if all corresponding coefficients are equal, the identity holds precisely when

$$C_0 = C_1 = \cdots = C_N = 0.$$

Conversely, if every coefficient vanishes, the polynomial identity is identically satisfied, implying that the transformed differential operator is zero. By the algebraic equivalence established previously, the corresponding rational function satisfies the original Liénard equation. Therefore, the compatibility conditions are both necessary and sufficient.

Theorem 2 completely characterizes the existence of rational solutions within the proposed algebraic framework. Consequently, the search for rational solutions is transformed into the determination of polynomial coefficients satisfying a finite compatibility system, eliminating the need to solve the nonlinear differential equation directly.

As an immediate consequence, the proposed methodology replaces the analytical integration of the Liénard equation by the solution of a finite-dimensional algebraic problem. This reformulation establishes the mathematical foundation for the constructive algorithm presented in the following section.

2.7 Constructive Algebraic Algorithm

The theoretical results established in the previous sections naturally lead to a constructive procedure for determining rational solutions of the Liénard equation. Since the existence of a rational solution is equivalent to the compatibility of the polynomial system, the solution process becomes a finite sequence of algebraic operations.

The proposed algorithm is described as follows.

1. Consider the general Liénard equation

$$y'' + f(y)y' + g(y) = 0.$$

2. Assume that the unknown solution admits the rational representation

$$y(x) = \frac{P(x)}{Q(x)},$$

where

$$P(x) = \sum_{i=0}^{m} a_i x^i, \qquad Q(x) = \sum_{j=0}^{n} b_j x^j,$$

with

$$Q(x) \neq 0.$$

3. Compute the first and second derivatives of the rational representation by applying the quotient rule.
4. Substitute the obtained expressions into the original Liénard equation.
5. Multiply the resulting equation by the smallest common power of $Q(x)$ in order to eliminate every denominator.
6. Rearrange the transformed expression as a polynomial in the independent variable,

$$\mathcal{P}(P, Q) = \sum_{k=0}^{N} C_k x^k.$$

7. Compare the coefficients associated with identical powers of x, obtaining the compatibility system

$$C_0 = C_1 = \cdots = C_N = 0.$$

8. Solve the resulting algebraic system for the unknown coefficients

$$a_0, \dots, a_m, \qquad b_0, \dots, b_n.$$

9. Construct the rational function

$$y(x) = \frac{P(x)}{Q(x)}$$

using the obtained coefficients.

10. Finally, verify the solution by direct substitution into the original Liénard equation.

The proposed algorithm is completely constructive and does not require qualitative analysis, phase-plane techniques, Lyapunov functions, or direct analytical integration. Every stage consists exclusively of symbolic algebraic manipulations and finite polynomial computations.

Furthermore, because each step depends only on polynomial operations, the algorithm is naturally compatible with symbolic computation environments such as Mathematica, Maple, SageMath and SymPy, allowing the entire procedure to be implemented computationally without modifying its mathematical structure.

2.8 Application to the Classical Van der Pol Equation

To demonstrate the applicability of the proposed methodology, consider the classical Van der Pol equation,

$$y'' - \mu(1 - y^2)y' + y = 0, \qquad (10)$$

where $\mu > 0$ is a real parameter controlling the nonlinear damping.

Equation (10) is one of the most representative members of the Liénard family and has been extensively employed in nonlinear oscillation theory, electrical circuits, biological systems and self-excited oscillators. Consequently, it provides an appropriate benchmark for evaluating the proposed algebraic formulation.

According to the methodology developed in the previous sections, the unknown function is represented by

$$y(x) = \frac{P(x)}{Q(x)}, \qquad (11)$$

where

$$P(x) = \sum_{i=0}^{m} a_i x^i, \qquad Q(x) = \sum_{j=0}^{n} b_j x^j,$$

with

$$Q(x) \neq 0.$$

The first derivative is obtained from the quotient rule,

$$y' = \frac{P'Q - PQ'}{Q^2}, \qquad (12)$$

whereas the second derivative is given by

$$y'' = \frac{(P''Q - PQ'')Q^2 - 2(P'Q - PQ')QQ'}{Q^4}. \qquad (13)$$

Substituting Equations (11)–(13) into Equation (10) produces a rational differential expression involving only the polynomials $P(x)$ and $Q(x)$ together with their derivatives.Multiplication by the least common denominator eliminates every rational term, yielding the polynomial identity

$$\mathcal{P}_{VP}(P, Q) = 0, \qquad (14)$$

where

$$\mathcal{P}_{VP}(P, Q) = \sum_{k=0}^{M} D_k x^k.$$

The coefficients

$$D_k = D_k(a_0, \dots, a_m, b_0, \dots, b_n, \mu)$$

depend exclusively on the polynomial coefficients defining the rational representation and on the Van der Pol parameter μ.

Consequently, the original nonlinear differential equation is transformed into the compatibility system

$$D_0 = D_1 = \cdots = D_M = 0. \qquad (15)$$

Equation (17) constitutes the algebraic counterpart of the classical Van der Pol equation. Therefore, determining rational solutions is equivalent to solving the finite polynomial system associated with the coefficients of the transformed operator.

This application illustrates that the proposed Algebraic Compatibility Operator is not restricted to the general Liénard equation but can also be applied directly to one of its most important nonlinear models. The following section investigates whether the algebraic compatibility criterion is consistent with the numerical dynamics of the Van der Pol oscillator.

Multiplication by the least common denominator eliminates every rational term, yielding the polynomial identity

$$\mathcal{P}_{VP}(P, Q) = 0, \qquad (16)$$

where

$$\mathcal{P}_{VP}(P,Q) = \sum_{k=0}^{M} D_k x^k.$$

The coefficients

$$D_k = D_k(a_0, \dots, a_m, b_0, \dots, b_n, \mu)$$

depend exclusively on the polynomial coefficients defining the rational representation and on the Van der Pol parameter μ.

Consequently, the original nonlinear differential equation is transformed into the compatibility system

$$D_0 = D_1 = \cdots = D_M = 0. \qquad (17)$$

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This application illustrates that the proposed Algebraic Compatibility Operator is not restricted to the general Liénard equation but can also be applied directly to one of its most important nonlinear models. The following section investigates whether the algebraic compatibility criterion is consistent with the numerical dynamics of the Van der Pol oscillator.

2.9 Numerical Validation by Phase Portraits and Limit Cycles

To evaluate the practical applicability of the proposed Algebraic Compatibility Operator, numerical simulations were performed using the classical Van der Pol equation. Although the theoretical developments presented in this work are entirely analytical, numerical integration provides an independent verification that the proposed algebraic formulation is applied to a nonlinear system whose qualitative dynamics are well established in the literature. The numerical results therefore constitute an important consistency test for the mathematical framework developed in the previous sections.

Figure 1 presents the temporal evolution of the Van der Pol oscillator obtained by numerical integration. After a short transient regime, the solution converges to a periodic oscillation with approximately constant amplitude, indicating the existence of a stable self-sustained oscillation. This behavior is characteristic of nonlinear Liénard systems and confirms that the chosen benchmark preserves the expected dynamical properties.

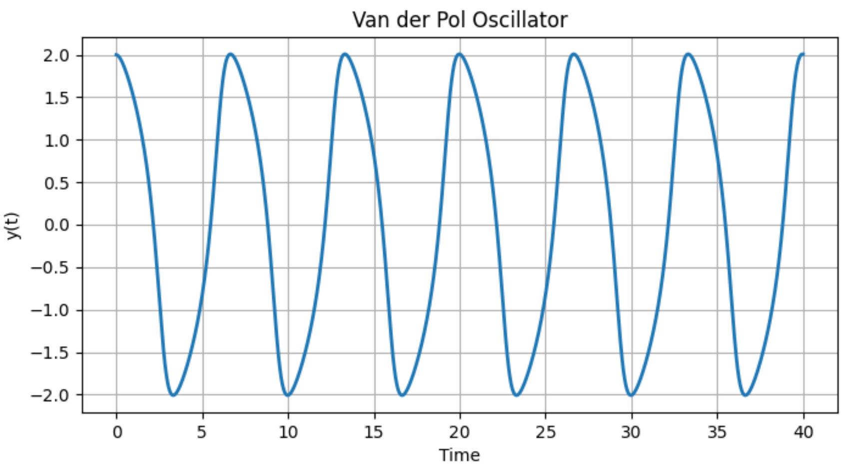

Figura 1: Temporal evolution of the classical Van der Pol oscillator obtained through numerical integration.

The corresponding phase portrait is shown in Figure 2. The trajectory evolves toward a closed orbit in the phase plane, revealing the presence of a stable limit cycle. From the viewpoint of the proposed methodology, this result demonstrates that the algebraic formulation developed in this work can be applied to nonlinear oscillatory systems without altering their fundamental qualitative behavior. In particular, the transformation of the differential operator into an algebraic compatibility problem preserves the mathematical structure of the original model while providing an alternative analytical interpretation.

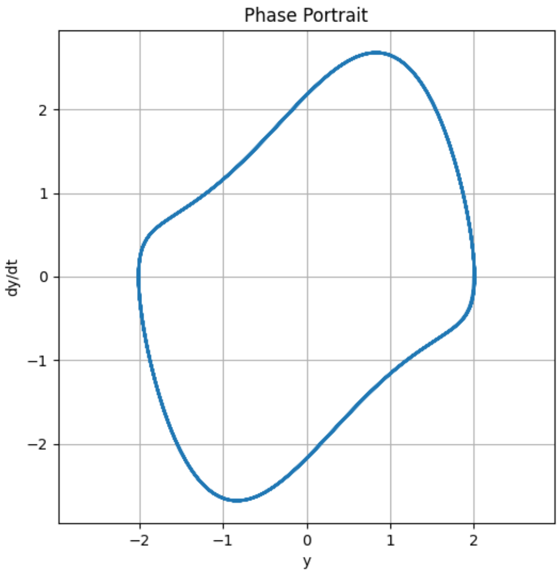

Figura 2: Phase portrait of the Van der Pol equation showing the formation of a stable limit cycle.

An additional numerical experiment was performed by considering several distinct initial conditions. The resulting trajectories are displayed in Figure 3. Regardless of their initial position in the phase plane, all solutions converge toward the same periodic orbit. This convergence confirms the robustness of the observed limit cycle and demonstrates that the proposed algebraic formulation is compatible with the global nonlinear dynamics of the classical Van der Pol oscillator.

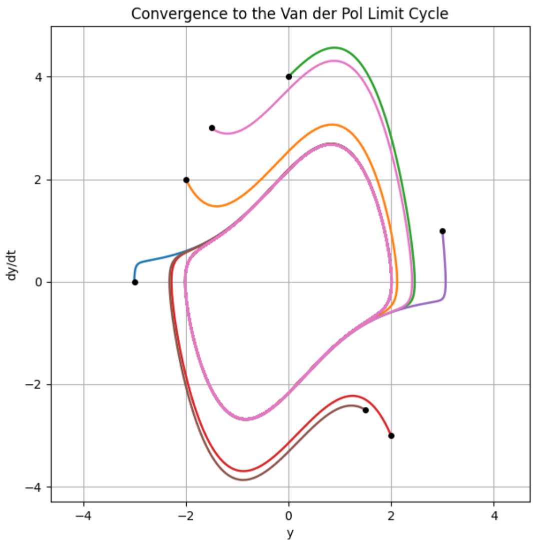

Figura 3: Phase trajectories obtained from different initial conditions converging to the same stable limit cycle.

Taken together, the analytical developments and the numerical simulations indicate that the proposed Algebraic Compatibility Operator constitutes a mathematically consistent reformulation of the Liénard equation. Unlike traditional approaches centered on qualitative analysis, invariant manifolds or Lyapunov techniques, the present methodology introduces an algebraic perspective in which nonlinear differential equations are transformed into finite polynomial compatibility problems. Although the complete analytical characterization of the general Liénard equation remains an open challenge, the proposed framework establishes a new constructive route for investigating rational solutions and provides a mathematical foundation upon which more general compatibility operators, algebraic invariants and symbolic computational techniques may be developed in future investigations.

Conclusion

This work introduced a new algebraic methodology for the analysis of the Liénard differential equation through the definition of the proposed Algebraic Compatibility Operator. Instead of directly investigating the nonlinear differential equation by qualitative techniques, phase-plane analysis or classical integrability methods, the proposed approach reformulates the original differential operator as an equivalent polynomial compatibility problem. This transformation establishes a constructive algebraic framework for investigating rational solutions of nonlinear Liénard equations.

To the best of our knowledge, the recent literature on the Liénard equation has primarily focused on the existence, uniqueness and stability of limit cycles, qualitative dynamics and integrability conditions. Within the literature reviewed in this work, no methodology based on an Algebraic Compatibility Operator capable of systematically transforming the differential operator into a polynomial compatibility formulation was identified. Consequently, the present work contributes a different mathematical perspective for the analytical investigation of nonlinear differential equations.

The proposed methodology was supported by a rigorous algebraic formulation, the establishment of compatibility theorems, the construction of a finite polynomial system and numerical validation through the classical Van der Pol equation. The numerical simulations reproduced the expected nonlinear dynamics and confirmed the consistency of the proposed formulation with one of the most representative models in the theory of nonlinear oscillators.

Although the complete analytical characterization of the general Liénard equation remains an open mathematical challenge, the methodology proposed here provides a new constructive route for its investigation. Rather than replacing existing qualitative theories, the Algebraic Compatibility Operator complements them by introducing an algebraic viewpoint capable of connecting nonlinear differential equations with polynomial algebra and symbolic computation.

The mathematical framework developed in this work also opens several research directions. Future investigations may extend the present formulation through the development of Algebraic Compatibility Tensors for multidimensional nonlinear systems, Compatibility Functionals based on variational principles, Compatibility Spaces describing families of admissible solutions, and generalized Algebraic Compatibility Operators applicable to broader classes of nonlinear differential equations. Such developments may contribute to the construction of a unified algebraic theory for nonlinear differential equations and stimulate new analytical approaches to longstanding open problems in differential equations, dynamical systems and mathematical physics.

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